On a special center of spiral similarity

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Abstract

In this paper, we briefly discuss some nice properties and a particular configuration of spiral similarity that is useful for problem solving. We look into several example problems from recent years and provide various exercises to the reader.

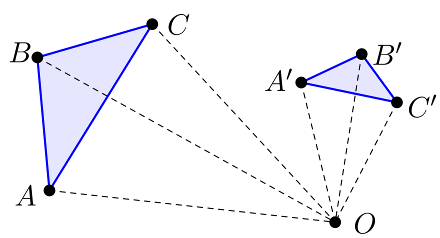
I acknowledge my debt to Cinthya Porras, Mauricio Rodríguez and Evan Chen for their valuable comments which helped a lot to improve this handout.

1. What is a spiral similarity?

Before discussing the core of this article, we revisit the setting and known properties of spiral similarity. First, let us describe what a spiral similarity consists of.

Definition 1.

A *spiral similarity* centered at a point O is a composition of a dilation and a rotation with respect to O.



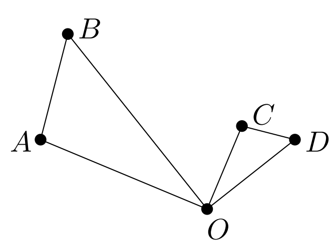
A spiral similarity about O which sends $\triangle ABC$ to $\triangle A'B'C'$.

2. Useful facts

For our purposes, it is enough to inquire about spiral similarity for segments. We deal with its uniqueness and existence before exploiting other crucial facts, what we effectively carry out as follows.

Lemma 1.

Given four points A, B, C and D on a same plane such that $\overline{AB}$ and $\overline{CD}$ are two distinct segments and quadrilateral $ABCD$ is not a parallelogram, there exists a unique spiral similarity carrying $\overline{AB}$ to $\overline{CD}$.



First proof.[1] Let $a,\ b,\ c,\ d,\ x_0$ be the corresponding complex numbers for $A,\ B,\ C,\ D$ and O. A spiral similarity has the form $\mathcal{T}(x) = x_0 + \alpha(x - x_0)$, where $|\alpha|$ is the dilation factor and $\arg\alpha$ the angle of rotation. Clearly,

$$\alpha = \frac{\mathcal{T}(a) - \mathcal{T}(b)}{a - b} = \frac{c - d}{a - b}$$

and because $\mathcal{T}(a) = x_0 + \alpha(a - x_0) = c$, it is straightforward to conclude that,

$$x_0 = \frac{ad - bc}{a + d - b - c}$$

Since $ABCD$ is not a parallelogram and $A \neq B$, we get $a + d \neq b + c$ and $a \neq b$, thus x_0 and α are well-defined. We have obtained exactly one solution for both x_0 and α, so we infer that indeed there is a unique spiral similarity mapping $\overline{AB}$ to $\overline{CD}$. □

Second proof.[2] We have $\triangle AOB \sim \triangle COD$, so $\frac{a-x_0}{c-x_0} = \frac{b-x_0}{d-x_0}$, which implies $x_0 = \frac{ad-bc}{a+d-b-c}$. Again, $a + d - b - c \neq 0$, hence O exists and is exactly determined by $A,\ B,\ C$ and D, therefore, such a spiral similarity is unique. □

Because of the previous result, it is okay to say *the* spiral similarity instead of *a* spiral similarity, since we now know it is unique. At this point, devising a way to construct the spiral similarity center would be illuminating. The following result comes to our rescue.

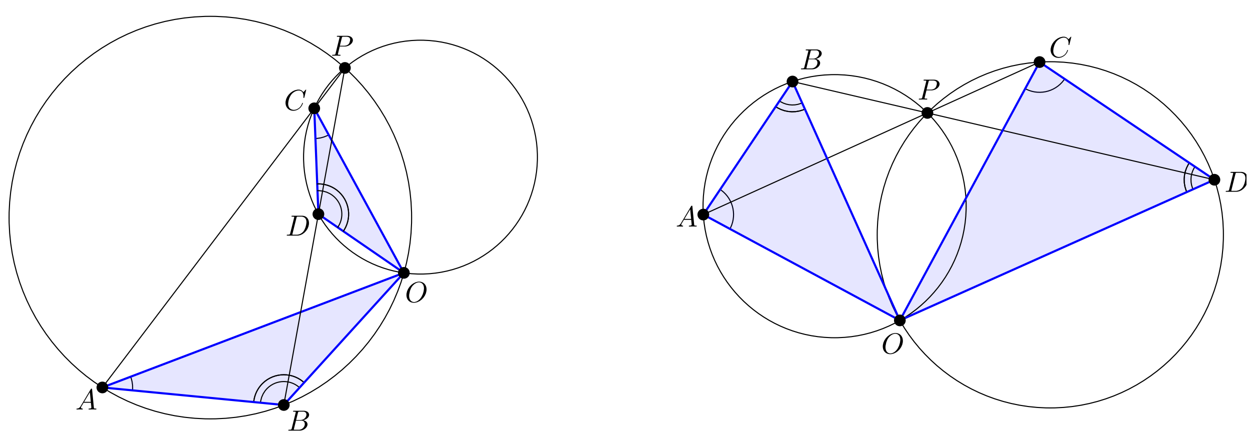
Lemma 2 (Extremely useful)**.**

Define P to be the intersection of $\overline{AC}$ and $\overline{BD}$ (i.e. $P = \overline{AC} \cap \overline{BD}$). Circles (ABP) and (CDP) meet at O ($O \neq P$); thus, O is the center of the spiral similarity which takes $\overline{AB}$ to $\overline{CD}$.

Proof. From $O \neq P$, it is clear that $ABCD$ cannot be a parallelogram. Notice that there are several possible spiral similarity configurations, so we use *directed angles*[3]. In fact, we observe,

$$\measuredangle OAB = \measuredangle OPD = \measuredangle OCD; \qquad \measuredangle ABO = \measuredangle APO = \measuredangle CDO$$

Therefore, triangles AOB and COD are *directly* similar. □



[1] A slightly different approach using complex numbers can be seen in Zhao (2010), pp. 3.
[2] See Chen (2016), pp. 196.
[3] We denote by $\measuredangle ABC$ the angle by which we rotate AB anticlockwise to make it parallel to BC. All directed angles are considered modulo 180°.

Moreover, observe that $\measuredangle COA = \measuredangle DOB$ and $\frac{AO}{CO} = \frac{OB}{OD}$, hence $\triangle AOC \sim \triangle BOD$ having the same orientation; then,

Lemma 3.

If O is the center of the spiral similarity sending $\overline{AB}$ to $\overline{CD}$, it is also the center of the spiral similarity taking $\overline{AC}$ to $\overline{BD}$.

This previous fact is super important! It turns out that spiral similarities always come in pairs. You must decide which pair of segments is more suitable to come up with a solution. Furthermore, it is worth mentioning that if X and Y are corresponding points in $\triangle AOB$ and $\triangle COD$, we would have $\triangle AOX \sim \triangle COY$ and $\triangle XOB \sim \triangle YOD$, so O carries $\overline{AX}$ to $\overline{CY}$ and $\overline{BX}$ to $\overline{DY}$.

3. What about $B = C$?

A pretty interesting situation arises when $B = C$ or $A = D$. Hereafter, assume the first equality and let us focus our attention on the following property.

Lemma 4.

Let ABD be a triangle and E the second point of intersection of the $B-$symmedian with its circumcircle. Then M, the midpoint of the chord BE, is the center of spiral similarity which maps $\overline{AB}$ to $\overline{BD}$.

Proof. Line BE is a symmedian of $\triangle ABD$, so AD is a symmedian of $\triangle EAB$ and $\triangle EDB$, therefore,

$$\angle MAB = \angle DAE = \angle DBE = \angle DBM; \qquad \angle BDM = \angle ADE = \angle ABE = \angle ABM$$

whence, $\triangle AMB \sim \triangle AED \sim \triangle BMD$ in the same orientation. Evidently, this special center of spiral similarity inherits the aforesaid uniqueness. The result follows. $\square$

Because BE is also a symmedian of $\triangle AED$, we can show along the same lines that $\triangle AME$, $\triangle ABD$ and $\triangle EMD$ are pairwise directly similar; thus, M takes $\overline{AE}$ to $\overline{ED}$. In summary, we have deduced the following results.

Lemma 5.

Let AB and BD be two segments in such a way A, B and D are not collinear. Let M be the center of spiral similarity which sends $\overline{AB}$ to $\overline{BD}$ and E the symmetric of B with respect to M. We have:

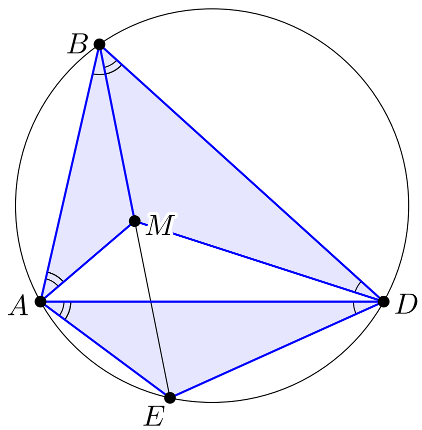
1. ME is the $B-$symmedian of $\triangle ABD$ (hence, it is the $E-$symmedian of $\triangle AED$).

2. Quadrilateral $ABDE$ is cyclic.

3. M sends segment AE to segment ED.

4. A takes $\overline{BM}$ to $\overline{DE}$ and D takes $\overline{BM}$ to $\overline{AE}$ (hence, according to **lemma 3**, A carries $\overline{BD}$ to $\overline{ME}$ and D carries $\overline{BA}$ to $\overline{ME}$).

Despite their simplicity, lemmas **4** and **5** are useful facts which shorten and lead to elegant solutions. Do not be fooled while attempting a problem with a hidden symmedian and its midpoint. Keep it in mind!



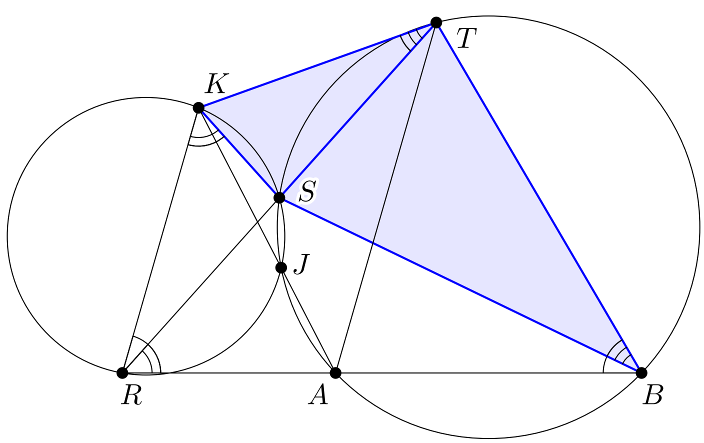
The special center of spiral similarity M

4. Example problems

In most problems, recognizing and using spiral similarity represent intermediate steps to show the required assertions, providing meaningful and important ideas, though. Let us solve some recent problems using the lemmas we have discussed so far.

Example 1.

(IMO 2017, P4) Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets again Ω at K. Prove that the line KT is tangent to Γ.



Proof. Let $B = \overline{RA} \cap \Gamma,\ A \neq B$. Notice that,

$$\angle SBR = \angle SBA = \angle SJK = \angle SRK$$

and since ℓ is tangent to Ω we obtain $\angle BRS = \angle RKS$, so $\triangle SKR \sim \triangle SRB$, thereby, S is the center of spiral similarity sending $\overline{KR}$ to $\overline{RB}$, but **lemma 5** tells us S takes $\overline{KT}$ to $\overline{TB}$, thus $\angle STK = \angle SBT$, as desired. □

Example 2.

(IMO 2014, P4) Points P and Q lie on side BC of acute-angled triangle ABC so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that the lines BM and CN intersect on the circumcircle of triangle ABC.

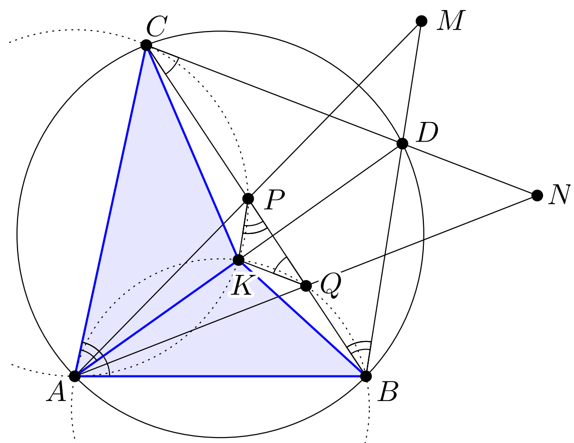
Proof. Let K be the center of spiral similarity carrying $\overline{BA}$ to $\overline{AC}$ and D the reflection of A respect to K. Recalling **lemma 5**, we know D lies on the circle (ABC). We will show that $D = \overline{BM} \cap \overline{CN}$, which clearly solves the problem. Note that $KQ \parallel DN$ and $KP \parallel DM$, so it suffices to prove that $KQ \parallel CD$ and $KP \parallel BD$. Observe that,

$$\angle QAK = \angle BAK - \angle BAQ = \angle BAC - \angle KAC - (\angle BAC - \angle QAC) = \angle CBA - \angle KBA = \angle QBK$$

therefore, quadrilateral $KQBA$ is cyclic. Similarly, we can show that $KPCA$ is cyclic as well. Hence,

$$\angle CQK = \angle BAK = \angle BAD = \angle BCD = \angle QCD; \qquad \angle KPB = \angle KAC = \angle DAC = \angle DBC = \angle DBQ$$

which readily give us the required parallelisms. □



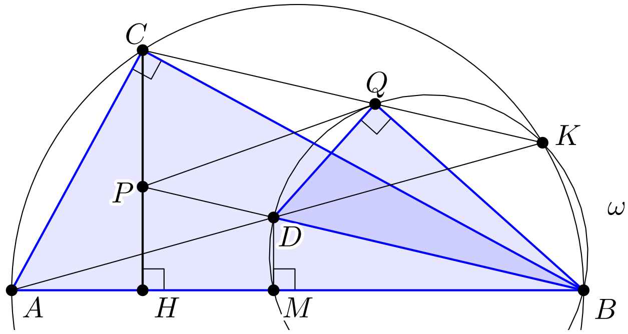
Example 3.

(IMO 2015 SL, G3) Let ABC be a triangle with $\angle C = 90°$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

Proof. Let M be a point on AB so that $\angle BMD = 90°$, so $CH \parallel DM$. Since CH bisects $\overline{AD}$, CH is the perpendicular bisector of $\overline{AM}$, so $AH = HM$. Note that,

$$\frac{PQ^2}{PD^2} = \frac{PB}{PD} = \frac{HB}{HM} = \frac{HB}{AH} = \frac{CB^2}{BA} : \frac{CA^2}{AB} = \frac{CB^2}{CA^2}$$

thus, $\frac{PQ}{PD} = \frac{CB}{CA}$. But $\triangle PQD \sim \triangle PBQ$, then $\frac{CB}{CA} = \frac{PQ}{PD} = \frac{QB}{QD}$ and we know $\angle ACB = 90° = \angle DQB$, whence $\triangle ACB \sim \triangle DQB$, thereby B is the center of the spiral similarity mapping $\overline{AC}$ to $\overline{DQ}$; hence, invoking **lemma 2** we infer that CQ and AD meet each other at the second point of intersection of ω and the circle (ABC), say K. The conclusion follows. □



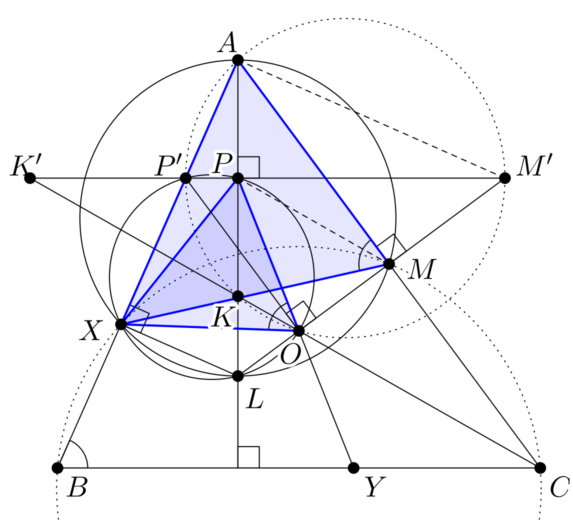
Example 4.

(IGO 2017, Advanced Level, P3) Let O be the circumcenter of triangle ABC. Line CO intersects the altitude through A at point K. Let P, M be the midpoints of $\overline{AK}$, $\overline{AC}$, respectively. If PO intersects BC at Y, and the circumcircle of triangle BCM meets AB at X, prove that $BXOY$ is cyclic.

Proof. We use directed angles. Let ℓ be the line parallel to BC through P, $P' = \overline{AB} \cap \ell$, $K' = \overline{CO} \cap \ell$, $M' = \overline{OM} \cap \ell$, $L = \overline{OM} \cap \overline{AK}$. Clearly, $\measuredangle MXA = \measuredangle ACB = \measuredangle MLA$, so $AXLM$ is cyclic, which gives $\measuredangle LXP' = 90°$. Observe that $\measuredangle M'PA = 90° = \measuredangle M'MA$, then $AM'MP$ is cyclic as well. Taking into account that $MP \parallel CK'$, we obtain,

$$\measuredangle MAM' = \measuredangle MPM' = \measuredangle CK'M' = \measuredangle OCB = 90° - \measuredangle BAC$$

therefore, $\measuredangle P'AM' = 90°$; since K is the reflection of A across ℓ, $AP'KM'$ must be a cyclic kite. Notice that $\measuredangle M'P'A = \measuredangle CBA = \measuredangle MOA = \measuredangle M'OA$, which implies that $AP'OM'$ is cyclic. We infer A, K and O lie on a circle with diameter $\overline{P'M'}$, hence $\measuredangle MOP' = 90° = \measuredangle LXP'$, thus $P'XLO$ is cyclic. Furthermore, we have $\measuredangle M'PL = 90°$ and we conclude that P lies on the circle $(P'XLO)$. Finally, we have $L = (PXO) \cap (AMX)$, $L \neq X$ and $L = \overline{AP} \cap \overline{MO}$, then, according to **lemma 2**, X is the center of spiral similarity taking $\overline{PO}$ to $\overline{AM}$, whence $\triangle PXO \sim \triangle AXM$, so $\measuredangle POX = \measuredangle AMX = \measuredangle CBA = \measuredangle YBX$, which completes the solution. ☐



We end this section by having a look at the following nice and hard problem.

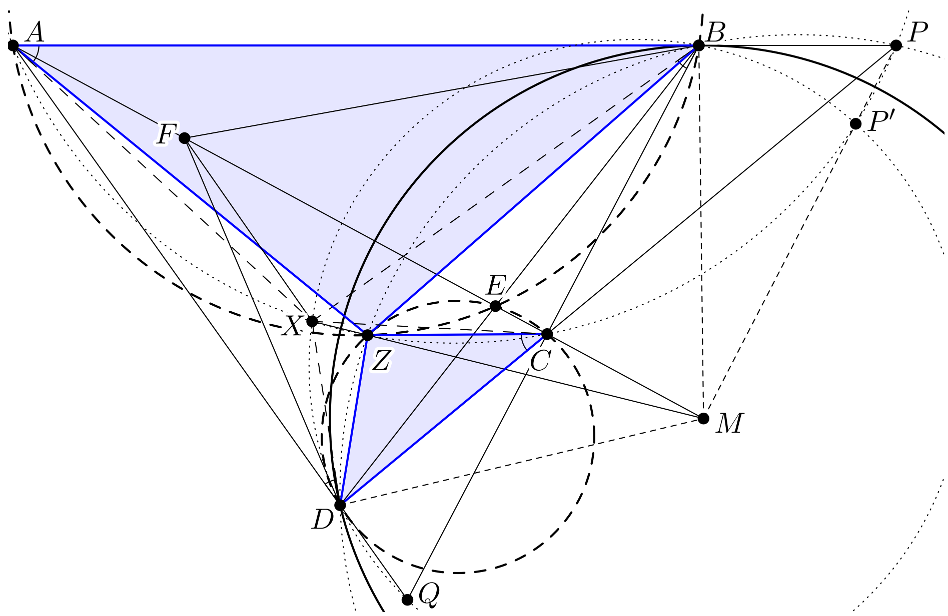
Example 5.

(IMO 2018, P6) A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that,

$$\angle XAB = \angle XCD \qquad \text{and} \qquad \angle XBC = \angle XDA$$

Prove that $\angle BXA + \angle DXC = 180^\circ$.

Proof. Let $P = \overline{AB} \cap \overline{CD}$, $Q = \overline{AD} \cap \overline{BC}$. From the angle conditions, it is straightforward to infer that X is the intersection point of circles (ACP) and (BDQ) located inside $ABCD$. Construct points $E = \overline{AC} \cap \overline{BD}$ and $Z = (BAE) \cap (CDE)$, $Z \neq E$. By **lemma 2**, Z sends $\overline{AB}$ to $\overline{CD}$ and according to **lemma 3**, it also maps $\overline{AC}$ to $\overline{BD}$, therefore, quadrilaterals $AZCP$ and $DZBP$ are cyclic.



Since $\frac{AB}{BC} = \frac{DA}{CD}$, we learn D lies on the B-Apollonius circle of $\triangle ABC$, say ω. Let M be its center and let X' be the inverse of Z with respect to ω. Note that,

$$MZ \cdot MX' = MD^2 = MC \cdot MA$$

thus, X' lies on $(APCZ)$ and A is the inverse of C with respect to ω. Let F be the inverse of E with respect to ω; evidently, it lies on $\overline{AC}$. Because $ABEZ$ and $CEZD$ are inscribed quadrilaterals, $CBFX'$ and $AFX'D$ must be cyclic, too; hereby, using directed angles we deduce that,

$$\measuredangle DX'B = \measuredangle FX'B + \measuredangle DX'F = \measuredangle FCB + \measuredangle DAF = \measuredangle FCB + \measuredangle DAC = \measuredangle ACQ + \measuredangle QAC = \measuredangle DQC$$

i.e. X' lies on (BDQ). In other words, X' is a common point for both $(AZCP)$ and (BDQ). Define P' to be the inverse of P respect to ω. Since $(AZCP)$ is orthogonal to ω, P' is on $(AZCP)$. Because $DZBP$ is cyclic, quadrilateral $DX'BP'$ is cyclic as well, which implies that Q, D, X', B, P' all lie on a same circumference; then, P' is the second intersection point of circles $(AZCP)$ and (QDB). We conclude that $X' = X$ or $X' = P'$. If $X' = P'$ we would have $Z = P$, which is absurd; hence, $X = X'$.
Finally, observe that,

$$\measuredangle AXD = \measuredangle AFD = \measuredangle MFD = \measuredangle BFC = \measuredangle BXC$$

whence, $\angle AXD + \angle CXB = 180^\circ$, which means $\angle BXA + \angle DXC = 180^\circ$. We are done! □

5. Practice problems

We have invoked the basic spiral similarity machinery and a particular configuration through various examples and their solutions. Now, it is time to try some exercises by your own. Naturally, geometry problems can be solved through different approaches, but the reader is aimed to tackle the exercises below using the results here addressed. These problems are attempted to be arranged in order of difficulty; nevertheless, it is hard to judge this accurately.

Problem 1. (Japan MO Finals 2018, P2) Given a scalene triangle ABC, let D and E be points on segments AB and AC, respectively, so that $CA = CD,\ BA = BE$. Let ω be the circumcircle of triangle ADE and P the reflection of A across BC. Lines PD and PE meet ω again at X and Y, respectively. Prove that BX and CY intersect on ω.

Problem 2. (Greece National MO 2018, P2) Let ABC be an acute-angled triangle with $AB < AC < BC$ and Γ its circumcircle. Let D and E be points in the shorter arcs AC and AB of Γ, respectively. Let K be the intersection point of BD and CE, and N the second common point of the circumcircles of triangles BKE and CKD. Prove that $A,\ K$ and N are collinear if and only if K belongs to the A-symmedian of ABC.

Problem 3. (Russian Sharygin GO 2016, Correspondence Round, P11) Restore a triangle ABC by vertex B, the centroid and the common point of the symmedian from B with the circumcircle.

Problem 4. (Romanian Master of Mathematics 2018, P1) Let $ABCD$ be a cyclic quadrangle and let P be a point on the side AB. The diagonal AC crosses the segment DP at Q. The parallel through P to CD crosses the extension of the side BC beyond B at K, and the parallel through Q to BD crosses the extension of the side BC beyond B at L. Prove that the circumcircles of the triangles BKP and CLQ are tangent.

Problem 5. (APMO 2017, P2) Let ABC be a triangle with $AB < AC$. Let D be the intersection point of the internal bisector of angle $\angle BAC$ and the circumcircle of ABC. Let Z be the intersection point of the perpendicular bisector of AC with the external bisector of angle $\angle BAC$. Prove that the midpoint of the segment AB lies on the circumcircle of triangle ADZ.

Problem 6. (Korea MO Final Round 2018, P2) Triangle ABC satisfies $\angle ABC < \angle BCA < \angle CAB < \angle 90^\circ$. Point O is the circumcenter of triangle ABC, and K is the reflection of O across BC. Points $D,\ E$ are the feet of the perpendicular lines from K to $AB,\ AC$, respectively. Line DE meets BC at P, and a circle with diameter AK meets the circumcircle of triangle ABC at Q $(Q \neq A)$. If PQ cuts the perpendicular bisector of $\overline{BC}$ at S, then prove that S lies on the circle with diameter $\overline{AK}$.

Problem 7. (Romania TST 2016, P1) Two circles ω_1 and ω_2 centered at O_1 and O_2, respectively, meet at points A and B. A line through B meet ω_1 again at C, and ω_2 again at D. The tangents to ω_1 and ω_2 at C and D, respectively, meet at E, and the line AE meets the circle ω through $A,\ O_1,\ O_2$ again at F. Prove that the length of $\overline{EF}$ is equal to the diameter of ω.

Problem 8. (Russian Sharygin GO 2016, Correspondence Round, P20) The incircle ω of a triangle ABC touches $BC,\ AC$ and AB at points $A_0,\ B_0$ and C_0 respectively. The bisectors of angles B and C meet the perpendicular bisector of $\overline{AA_0}$ at points Q and P, respectively. Prove that PC_0 and QB_0 meet on ω.

Problem 9. (Russian Sharygin GO 2015, 10^{th} grade, P3) Let $A_1,\ B_1$ and C_1 be the midpoints of sides $BC,\ CA$ and AB of triangle ABC. Points B_2 and C_2 are the midpoints of $\overline{BA_1}$ and $\overline{CA_1}$, respectively. Point B_3 is symmetric to C_1 with respect to B, and C_3 is symmetric to B_1 with respect to C. Prove that one of the common points of circles BB_2B_3 and CC_2C_3 lies on the circumcircle of triangle ABC.

Problem 10. (Iran MO 2016, 3^{rd} Round) Let ABC be an arbitrary triangle. Let E and F be two points on sides AB and AC, respectively such that their distances to the midpoint of BC are equal. Circumcircles of triangles ABC and AEF intersect at point P ($P \neq A$). The tangents from E and F to the circumcircle of AEF meet at a point K. Prove that $\angle KPA = 90°$.

Problem 11. (Iran TST 2018, Test 2, P5) Let ω be the circumcircle of an isosceles triangle ABC ($AB = AC$). Points P and Q lie on ω and BC, respectively, such that $AP = AQ$. Lines AP and BC intersect each other at R. Prove that the tangents from B and C to the incircle of triangle AQR (different to BC) concur on ω.

Problem 12. (Iberoamerican MO 2017 SL, G4) Let ABC be an acute-angled triangle with $AB > AC$, circumcircle Γ and M the midpoint of $\overline{BC}$. A point N lies inside of ABC so that $DE \perp AM$, where D and E are the feet of the altitudes from N to AB and AC, respectively. The circumcircle of ADE meets Γ again at L and K is the intersection point of AL and DE. Line AN meets Γ again at F. If N coincides with the midpoint of $\overline{AF}$, show that $KA = KF$.

Problem 13. (USAMO 2017, P3) Let ABC be a scalene triangle with circumcircle Ω and incenter I. Ray AI meets $\overline{BC}$ at D and meets Ω again at M; the circle with diameter $\overline{DM}$ cuts Ω again at K. Lines MK and BC meet at S, and N is the midpoint of $\overline{IS}$. The circumcircles of $\triangle KID$ and $\triangle MAN$ intersect at points L_1 and L_2. Prove that Ω passes through the midpoint of either $\overline{IL_1}$ or $\overline{IL_2}$.

Problem 14. (Iran TST 2010, P5) Circles ω_1 and ω_2 intersect at P and K. Points X and Y lie on ω_1 and ω_2, respectively, so that XY is tangent externally to both circles and XY is closer to P than K. Line XP meets ω_2 again at C and line YP meets ω_1 again at B. Lines BX and CY intersect at A. Prove that if Q is the second intersection point of the circumcircles of triangles ABC and AXY, then $\angle QXA = \angle QKP$.

Problem 15. (IMO 2016 SL, G5) Let D be the foot of perpendicular from A to the Euler line (the line passing through the circumcenter and the orthocenter) of an acute scalene triangle ABC. A circle ω with center S passes through A and D, and it intersects sides AB and AC at X and Y respectively. Let P be the foot of the altitude from A to BC, and let M be the midpoint of BC. Prove that the circumcenter of triangle XSY is equidistant from P and M.

Problem 16. (European Mathematical Cup 2016, P4) Let $C_1,\ C_2$ be circles intersecting in $X,\ Y$. Let $A,\ D$ be points on C_1 and $B,\ C$ on C_2 such that $A,\ X,\ C$ are collinear and $D,\ X,\ B$ are collinear. The tangent to circle C_1 at D intersects BC and the tangent to C_2 at B in $P,\ R$ respectively. The tangent to C_2 at C intersects AD and the tangent to C_1 at A in $Q,\ S$, respectively. Let W be the intersection of AD with the tangent to C_2 at B and Z the intersection of BC with the tangent to C_1 at A. Prove that the circumcircles of triangles $YWZ,\ RSY$ and PQY have two points in common, or are tangent in the same point.

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